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About the Solution in Closed Form of Generalized Four-Element Riemann Boundary Value Problems

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Abstract

The generalized four-element Riemann boundary value problems

$$a(t)\varphi^{+}(t)+b(t)\overline{\varphi^{+}(t)}=c(t)\varphi^{-}(t)+d(t)\overline{\varphi^{-}(t)}+f(t),\quad |t|=1,\quad \varphi^{-}(\infty)=0$$

is investigated in the class of piecewise analytic functions. When $a(t)=-\overline{b(t)}\neq 0$, $c(t)\neq -\overline{d(t)}$ are satisfied, we discuss it's noether theory, stability, and solvability theory, then the closed form of the solution of problem above can be established

Keywords: Generalized four-element Riemann boundary value problems, Markushevich problem, Close solution

1 Introduction

Let L is a simple closed Lyapunov curve dividing the closed complex plane into the interior part D^{+} and exterior part D^{-}, $0\in D^{+}$. Find function $\varphi^{+}(z)$ and

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$\varphi^-(z)$ analytic in D^+ and D^-, respectively, satisfying the condition

$$a(t)\varphi^+(t) + b(t)\overline{\varphi^+(t)} = c(t)\varphi^-(t) + d(t)\overline{\varphi^-(t)} + f(t), \quad t \in L \tag{1}$$

imposed on their boundary values on the contour L.

This problem is called generalized four-element Riemann boundary value problems [5]. And it should be noted that the problem in form (1) in case $a(t) = 1, b(t) = 0$ firstly was formulated in 1946 by A.I.Markushevich [6].

Many original works[1-4,7] have been devoted to the problem (1). And G.S.Litvinchuk [5] reviewed the survey of closely related results of problem (1), and then the noether theory, stability, and solvability theory were all mentioned.

In this article we shall obtain the constructive algorithm for solution of the problem (1), and when the conditions $a(t) = -\overline{b(t)} \neq 0$, $c(t) \neq -\overline{d(t)}$ are satisfied, the problem can be solved in a closed form.

2 The solution of problem (1) in a close form

Let the curve L be the unit circle, i.e. $D^+ = \{z : |z| < 1\}$, $D^- = \{z : |z| > 1\}$, and $a(t), b(t)$, $c(t)$, $d(t)$, $f(t)$ are given on L functions of Holder class. When the conditions $a(t) = -\overline{b(t)} \neq 0, c(t) \neq -\overline{d(t)}$ are satisfied, we shall solve the problem (1) under the condition $\varphi^-(\infty) = 0$.

2.1 Noetherity conditions, stable and degenerated properties

In this section, we first discuss the noetherian, stable and degenerated case of problem (1) under the assumption.

From conditions $a(t) = -\overline{b(t)} \neq 0$, $c(t) \neq -\overline{d(t)}$, then we have the inequality

$$\delta(t) = \overline{a(t)}c(t) - b(t)\overline{d(t)} \neq 0$$

According to reference [5], the problems (1) is said to be noetherian.

Let $\beta(t) = t[\overline{a(t)}d(t) - b(t)\overline{c(t)}]$. Firstly, let's introduce the solution of problem (1) under stable case or degenerated case

Lemma[5] If one of the following conditions holds:

1) $(|a(t)| - |b(t)|)(|c(t)| - |d(t)|) > 0$, (stable case)

2) $|c(t)| = |d(t)| > 0, \quad |a(t)| - |b(t)| \neq 0, \quad \{\arg \beta(t)\}_L \geq -|\{\arg \delta(t)\}_L|$, (degenerated case)

3) $|a(t)| = |b(t)| > 0, \quad |c(t)| - |d(t)| \neq 0, \quad \{\arg \beta(t)\}_L \leq |\{\arg \delta(t)\}_L|$, (degenerated case)

4) $|a(t)| = |b(t)| > 0, \quad |c(t)| - |d(t)| > 0, \quad |\{\arg \beta(t)\}_L| \leq |\{\arg \delta(t)\}_L|$. (degenerated case)

Then the number l of linearly independent solutions and the number p of linearly independent solvability conditions of the generalized four-element Riemann boundary value problems (1) for one pair of functions are given by

$$l = \max(0, \frac{1}{\pi}\{\arg\delta(t)\}_L), \quad p = \max(0, -\frac{1}{\pi}\{\arg\delta(t)\}_L)$$

Notice that, if we impose the conditions $a(t) = -\overline{b(t)} \neq 0$, $c(t) \neq -\overline{d(t)}$, the following identities can be directly verified

$$\frac{1}{2\pi}\{\arg\beta(t)\}_L = 1 - \frac{1}{2\pi}\{\arg\delta(t)\}_L$$

So according to the reference [5], the equality above means that the problem (1) under the conditions $a(t) = -\overline{b(t)} \neq 0$, $c(t) \neq -\overline{d(t)}$ is not stable or degenerated.

2.2 The solution of problem (1)

Let's take conjugates on both sides of problem (1)

$$\overline{a(t)}\,\overline{\varphi^+(t)} + \overline{b(t)}\varphi^+(t) = \overline{c(t)}\,\overline{\varphi^-(t)} + \overline{d(t)}\varphi^-(t) + \overline{f(t)}, \quad t \in L \tag{2}$$

Then by (1) and (2), and the conditions $a(t) = -\overline{b(t)} \neq 0$, $c(t) \neq -\overline{d(t)}$, then we have

$$\varphi^-(t) = -\frac{\overline{c(t)} + d(t)}{c(t) + \overline{d(t)}}\overline{\varphi^-(t)} - \frac{\overline{f(t)} + f(t)}{c(t) + \overline{d(t)}}, \quad t \in L \tag{3}$$

If we denote $G(t) = -\dfrac{\overline{c(t)} + d(t)}{c(t) + \overline{d(t)}}$, $g(t) = -\dfrac{\overline{f(t)} + f(t)}{c(t) + \overline{d(t)}}$, then $G(t), g(t) \in H_\mu(L)$. And it implies that

$$\begin{cases} |G(t)| = 1 \\ G(t)\overline{g(t)} + g(t) = 1 \end{cases} \tag{4}$$

1) Find function $\varphi(z)$ in D^-

Denote $\kappa = \dfrac{1}{2\pi}\{\arg G(t)\}_L$. Obviously, κ is an even number, called the index of problem (3).

(1) Assume $\kappa = 2m < 0$, the problem (3) under $\varphi^-(\infty) = 0$ has its general solution

$$\varphi^-(z) = z^m X_0^-(z)\left\{\frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z}d\tau + \frac{ib}{2} + \sum_{j=0}^{-2m-2}\beta_j w_j(z)\right\} \tag{5}$$

where

$$\begin{cases} w_0(z) \equiv 1 \\ w_{2k-1}(z) = z^k + \dfrac{1}{2\pi i}\displaystyle\int_L \frac{\phi_{2k-1}(\tau)}{\tau - z} d\tau + i\frac{b_{2k-1}}{2} \\ w_{2k}(z) = iz^k + \dfrac{1}{2\pi i}\displaystyle\int_L \frac{\phi_{2k}(\tau)}{\tau - z} d\tau + i\frac{b_{2k}}{2} \end{cases} \tag{6}$$

$k = 1,2,...,-m-1$. And $\phi_{2k-1}(t), \phi_{2k}(t)$, $\phi(t)$ are the solutions of equation as follows

$$\begin{cases} \tilde{\Re}_+\phi_{2k-1} = -t^k + \overline{t^k} - ib_{2k-1} \\ \tilde{\Re}_+\phi_{2k} = -it^k - \overline{it^k} - ib_{2k} \\ \tilde{\Re}_+\phi = \dfrac{g(t)}{t^m X_0^-(t)} - ib \end{cases} \tag{7}$$

where $X_0^-(z)$ is a basic function, and it satisfies the boundary value condition

$$X_0^-(t) = \frac{\overline{t^m} G(t)}{t^m} \overline{X_0^-(t)} \tag{8}$$

$\tilde{\Re}_+\phi$ is a Fredholm integral equation

$$(\tilde{\Re}_+\phi)(t) \equiv -\phi(t) + \frac{1}{2\pi i}\int_L \left\{ \frac{\tau'}{\tau - t} - \frac{\overline{\tau'^2(\sigma)}}{\overline{\tau} - \overline{t}} \right\} \phi(\tau) d\tau = 0 \tag{9}$$

(2) $\kappa = 2m \geq 0$. In this case, the conditions of solvability of problem (3) under $\varphi^-(\infty) = 0$ are

$$\begin{aligned} & b = \mathrm{Im}\left\{ \frac{\displaystyle\int_L \frac{g(t)}{t^m \cdot X_0^-(t)} \psi(t) dt}{\displaystyle\int_L \psi(t) dt} \right\} = 0 \\ & \mathrm{Re}\int_L \phi(t) \cdot t^{j-1} dt = 0 \\ & \mathrm{Im}\int_L \phi(t) \cdot t^{j-1} dt = 0 \end{aligned} \tag{10}$$

$j = 1,2,..., m$, and $\psi(t)$ is a solution of the adjoint equation $\tilde{\Re}_+\psi = 0$.

If and only if they are fulfilled, the problem (3) has the unique solution as follows

$$\varphi^-(z) = z^m X_0^-(z) \left\{ \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - z} d\tau + \frac{ib}{2} \right\} \tag{11}$$

2) Find function $\varphi(z)$ in D^+

From (5), we have

$$\varphi^-(t) = t^m X_0^-(t)\left\{-\frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_L \frac{\phi(\tau)}{\tau - t}d\tau + \frac{ib}{2} + \sum_{j=0}^{-2m-2}\beta_j w_j(t)\right\} \tag{12}$$

Let $u(t) = \dfrac{c(t)\varphi^-(t) + d(t)\overline{\varphi^-(t)} + f(t)}{2}$, and by the assumption $a(t) = -\overline{b(t)} \neq 0$, then the problem (1) can be transferred to

$$\mathrm{Re}\{-ia(t)\varphi^+(t)\} = u(t), \quad t \in L \tag{13}$$

Obviously, only if $u(t)$ is a real function, i.e.

$$\mathrm{Im}\frac{c(t)\varphi^-(t) + d(t)\overline{\varphi^-(t)} + f(t)}{2} = 0 \tag{14}$$

the problem (13) is a Hilbert boundary value problem. Denote $\kappa_1 = \dfrac{1}{\pi}[\arg\overline{a(t)}]_L$, thus we have the following

(1) Assume $\kappa_1 \geq 0$, the problem (13) has its general solution

$$\varphi^+(z) = \varphi_0(z) + X(z)(C_0 z^{\kappa_1} + C_1 z^{\kappa_1 - 1} + ... + C_{\kappa_1}) \tag{15}$$

where $X(z)$ is the canonical function of problem (13), and

$$\begin{aligned}\varphi_0(z) = {} & \frac{X(z)}{2\pi}\left[\int_L \frac{u(t)dt}{a(t)X^+(t)(t-z)} + z^{\kappa_1}\int_L \frac{t^{-\kappa_1}u(t)dt}{a(t)X^+(t)(t-z)}\right] \\ & - \frac{z^{\kappa_1}X(z)}{2\pi}\int_L \frac{u(t)t^{-\kappa_1 - 1}dt}{a(t)X^+(t)}\end{aligned} \tag{16}$$

if and only if

$$C_0 = \overline{C_{\kappa_1}}, C_1 = \overline{C_{\kappa_1 - 1}}, ..., C_{\kappa_1/2} = \overline{C_{\kappa_1/2}} \tag{17}$$

where $C_0, C_1, ..., C_{\kappa_1}$ are arbitrary constants.

(2) If $\kappa_1 \leq -2$, and the following conditions are satisfied

$$\int_L \frac{t^k u(t)}{a(t)X^+(t)}dt = 0, \quad k = 0,1,...,-\frac{\kappa_1}{2} - 1 \tag{18}$$

the problem (13) is solvable and has a unique solution

$$\varphi^+(z) = \frac{X(z)}{\pi}\int_L \frac{u(t)dt}{a(t)X^+(t)(t-z)} \tag{19}$$

Thus, we get

Theorem Consider the following boundary value problem (1) under the condition $\varphi^-(\infty) = 0$. Let $\kappa = \dfrac{1}{2\pi}\left\{\arg\dfrac{\overline{c(t)} + d(t)}{c(t) + \overline{d(t)}}\right\}_L$, $\kappa_1 = \dfrac{1}{\pi}[\arg\overline{a(t)}]_L$. When some supplementary conditions $a(t) = -\overline{b(t)} \neq 0$, $c(t) \neq -\overline{d(t)}$ are satisfied, the sufficient and necessary conditions for solvability of the boundary value problem (1) are

① If $\kappa < 0$ and $\kappa_1 \geq 0$, then it has the solutions (5),(15) when and only when (14),(17) are fulfilled.
② If $\kappa < 0$ and $\kappa_1 \leq -2$, then it has the solutions (5),(19)when and only when (14),(18)are fulfilled.
③ If $\kappa \geq 0$ and $\kappa_1 \geq 0$, then it has the solutions (11),(15)when and only when (10),(14),(17) are fulfilled.
④ If $\kappa \geq 0$ and $\kappa_1 \leq -2$, then it has the solutions (11),(19) when and only when (10),(14),(18) are fulfilled.

Example. Let $D^+ = \{z : |z| < 1\}$, $D^- = \{z : |z| > 1\}$ and $L = \{t : |t| = 1\}$. It is required to find functions $\varphi^+(z)$ and $\varphi^-(z)$ analytic in D^+ and D^-, respectively, which vanishing on the infinity and satisfying on L the following boundary condition

$$\frac{1}{t}\varphi^+(t) - t\overline{\varphi^+(t)} = t^2\varphi^-(t) - (\frac{1}{t^2} + 3)\overline{\varphi^-(t)} + 4t - \frac{1}{t}, \quad t \in L \tag{20}$$

Solution. Here $a(t) = \frac{1}{t}$, $b(t) = -t$, $c(t) = t^2$, $d(t) = -\frac{1}{t^2} - 3$, $f(t) = 4t - \frac{1}{t}$, then we have that $a(t) + \overline{b(t)} = 0$, $c(t) + \overline{d(t)} = -3 \neq 0$. So from the theorem above, we get in this case

$$\kappa = 0, \quad \kappa_1 = 2, \quad u(t) = (\frac{1}{t} - t)i$$

Then the following functions will be the solution of the problem (20)

$$\varphi^-(z) = \frac{1}{z}$$

$$\varphi^+(z) = 2z^2 + cz$$

where c is arbitrary real constants.

3 Some special cases

1) If $a(t) = \overline{b(t)} \equiv 0$, the problem (1) can be written in the form

$$c(t)\varphi^-(t) + d(t)\overline{\varphi^-(t)} + f(t) = 0, \quad t \in L$$

Then it easily to see the problem can be written as follows

$$\mathrm{Re}\{[c(t) + \overline{d(t)}]\varphi^-(t)\} = -\mathrm{Re}\, f(t)$$

It is a Riemann-Hilbert outer problem, and then the closed form solution of problem (1) is obtained.

2) Similarly, if $c(t) = \overline{d(t)} \equiv 0$, the problem (1) can be written as

$$a(t)\varphi^{+}(t) + b(t)\overline{\varphi^{+}(t)} = f(t), \quad t \in L$$

Then it easily to see the problem can be written as follows

$$\mathrm{Re}\left\{[c(t) + \overline{d(t)}]\varphi^{+}(t)\right\} = \mathrm{Re}\, f(t), \quad t \in L$$

It is a Riemann-Hilbert inner problem, and then the closed form solution of problem (1) is obtained.

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